Dynamics and Kinetics. Exercises 7: Solutions

Problem 1

1. The most probable velocity v^* corresponds to the velocity in which $f_{\mathrm{MB}}(v)\, dv$ is maximum:

$$\frac{df_{\mathrm{MB}}}{dv} = 0 \quad \Leftrightarrow \quad 0 = \frac{d}{dv}\left(e^{-\frac{mv^2}{2k_BT}}v^2\right) = \underbrace{\left(-\frac{2mv}{2k_BT}v^2 + 2v\right)}_{=0} e^{-\frac{mv^2}{2k_BT}}$$

$$v^* = \sqrt{\frac{2k_BT}{m}}$$

2. Definition of root mean square velocity:

$$v_{\mathrm{rms}} = \langle v^2 \rangle^{1/2} \quad \to \quad \langle v^2 \rangle := \int_0^\infty v^2 f(v) dv = x$$

Instead of substitute and solve, use some tricks

$$x \overset{\text{Trick 1}}{=} \frac{\int_0^\infty v^2 f(v)dv}{\int_0^\infty f(v)dv} \overset{\text{Trick 2}}{=} \frac{I(4)}{I(2)} \overset{\text{Trick 3}}{=} \frac{1}{a}\frac{\Gamma(5/2)}{\Gamma(3/2)} = \frac{3}{2a} = \frac{3k_BT}{m}$$

$$v_{\mathrm{rms}} = \langle v^2 \rangle^{1/2} = x^{1/2} = \sqrt{\frac{3k_BT}{m}}$$

Trick 1: $\int_0^\infty f(v)dv = 1$

Trick 2: $v^2 f(v) \propto v^4 e^{-av^2}$ [resembling $I(4)$] and $f(v) \propto v^2 e^{-av^2}$ [resembling $I(2)$]

Trick 3: Make $a = \frac{m}{2k_BT}$ and use definition of $I(n)$ in terms of gamma functions

Problem 2

Main idea: probabilities should be conserved, i.e., $f(\varepsilon)d\varepsilon = f(v)\, dv$
Change of variables:

$$\varepsilon = \frac{mv^2}{2} \quad \Rightarrow \quad d\varepsilon = mv dv \quad \Rightarrow \quad v = \sqrt{\frac{2\varepsilon}{m}}$$

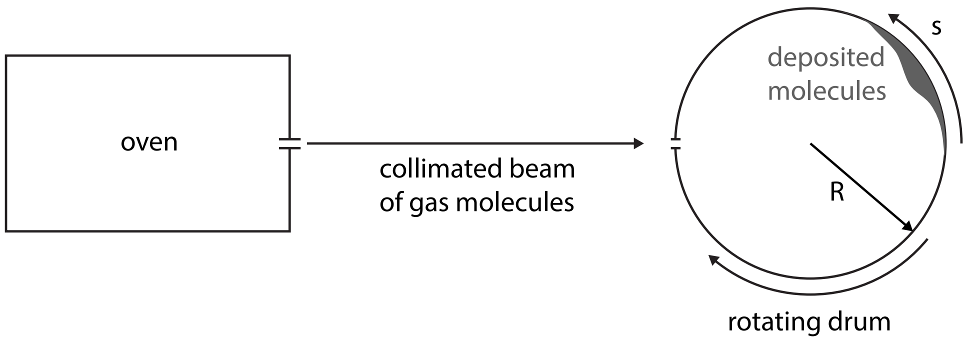
$$f(\varepsilon)d\varepsilon = f(v)dv = \left(\frac{m}{2\pi k_BT}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_BT}\right) 4\pi \underbrace{v^2 dv}_{= v \times v dv = \sqrt{\frac{2\varepsilon}{m}} \times \frac{d\varepsilon}{m}}$$

$$= \left(\frac{m}{2\pi k_BT}\right)^{3/2} \exp\left(-\frac{\varepsilon}{k_BT}\right) 4\pi \sqrt{\frac{2\varepsilon}{m}} \frac{d\varepsilon}{m}$$

$$= 2\pi \left(\pi k_BT\right)^{-3/2} \varepsilon^{1/2} \exp\left(-\frac{\varepsilon}{k_BT}\right) d\varepsilon$$

Problem 3

The figure below illustrates an experiment for measuring the Maxwell-Boltzmann distribution.



An oven at temperature T releases a narrow beam of gas molecules of mass m through a hole. The molecules strike a drum of radius R that rotates with frequency ν. The drum has a small opening through which gas molecules can enter into the drum when the opening passes through the beam of gas molecules. Because of the fast rotation of the drum, only a short pulse of gas molecules enters. Once these molecules reach the opposite wall of the drum, they stick to it.

a) Show that the flux of molecules of velocity u contained in a pulse that enters the drum is proportional to

$$u^3 e^{-\frac{mu^2}{2k_BT}} du$$

The flux of molecules of velocity u from the oven is proportional to u. In other words, the faster a molecule moves, the larger the number of molecules of that specific velocity that are contained in one pulse.

Moreover, the number of molecules of velocity u is proportional to the Maxwell-Boltzmann distribution and therefore to

$$u^2 e^{-\frac{mu^2}{2k_BT}} du$$

The flux is proportional to the product of both

$$I(u)du \propto u^3 e^{-\frac{mu^2}{2k_BT}} du$$

b) The molecules are deposited on the wall of the drum at a distance s from the point opposite the opening in the drum as indicated in the figure. Derive the distribution $I(s)ds$ of the deposited molecules. It is not necessary to normalize this distribution.

Molecules of velocity u will take a time $\frac{2R}{u}$ to traverse the drum. In this time, the drum will have rotated by a distance

$$s = 2\pi R\nu \cdot \frac{2R}{u} = \frac{4\pi R^2 \nu}{u}$$

With

$$du = -\frac{4\pi R^2 \nu}{s^2} ds$$

we find

$$I(s)ds \propto -\frac{(4\pi R^2 \nu)^4}{s^5} e^{-\frac{m\left(4\pi R^2 \nu\right)^2}{2k_B T s^2}} ds$$

Problem 4

Derive the speed distribution $F(u)du$ of a two-dimensional ideal gas.

Hint: Start from a one-dimensional velocity distribution to derive a two-dimensional distribution of the velocities, and then do a suitable coordinate transformation.

We begin with the one-dimensional distribution of the velocity

$$f(u_j) = \sqrt{\frac{m}{2\pi k_B T}} e^{-\frac{m u_j^2}{2k_B T}}$$

which gives us the two-dimensional velocity distribution

$$h(u_x, u_y) du_x du_y = \frac{m}{2\pi k_B T} e^{-\frac{m(u_x^2 + u_y^2)}{2k_B T}} du_x du_y$$

We do a coordinate transformation with

$$\begin{aligned} u^2 &= u_x^2 + u_y^2 \\ u_x &= u \cos\phi \\ u_y &= u \sin\phi \\ du_x du_y &= u du d\phi \end{aligned}$$

and obtain

$$\tilde{h}(u,\phi)dud\phi = \frac{m}{2\pi k_B T} u e^{-\frac{mu^2}{2k_B T}} dud\phi$$

which after integration over all angles becomes

$$F(u)du = \frac{m}{k_B T} u e^{-\frac{mu^2}{2k_B T}} du$$